# Continuity and Differentiability

**1.** When $x = \dfrac{1}{\left(2m+\frac{1}{2}\right)\pi}$ , $m \in \mathbb{Z}$. f(x) = 1. When $x = \dfrac{1}{\left(2m-\frac{1}{2}\right)\pi}$ , $m \in \mathbb{Z}$. f(x) = – 1.

Also, f(0) = 0. $\therefore$ f is not continuous at x = 0

Since f(x) is not continuous, it is not differentiable.

**2.** $f_n(x) = x^n \sin\frac{1}{x}$ , $x \neq 0$ and $f_n(0) = 0$. The function is obviously continuous at $x \neq 0$.

It is also continuous at x = 0 since for $n \in \mathbb{N}$ ,

$$\left|x^n \sin\frac{1}{x}\right| = \left|x^n\right|\left|\sin\frac{1}{x}\right| \le \left|x^n\right| \Rightarrow \lim_{x\to 0}\left|x^n \sin\frac{1}{x}\right| \le \lim_{x\to 0}\left|x^n\right| = 0 \Rightarrow \lim_{x\to 0} x^n \sin\frac{1}{x} = 0 \Rightarrow \lim_{x\to 0} f_n(x) = 0 = f_n(0)$$

If n = 1, $f_1'(0) = \lim_{h\to 0}\dfrac{f_1(0+h)-f_1(0)}{h} = \lim_{h\to 0}\dfrac{h\sin\frac{1}{h}-0}{h} = \lim_{h\to 0}\left(\sin\frac{1}{h}\right)$ , which is undefined (see Q.1)

$\therefore$ $f_1(x)$ is **not** differentiable at x = 0. It is differentiable at $x \in \mathbb{R}/\{0\}$.

If n > 1, then

$$f_n'(0) = \lim_{h\to 0}\frac{f_n(0+h)-f_n(0)}{h} = \lim_{h\to 0}\frac{h^n\sin\frac{1}{h}-0}{h} = \lim_{h\to 0}\left(h^{n-1}\sin\frac{1}{h}\right) = 0$$ , where n > 1.

When $x \neq 0$ , $f_n'(x) = \dfrac{d}{dx}\left(x^n \sin\frac{1}{x}\right) = nx^{n-1}\sin\frac{1}{x} - x^{n-2}\cos\frac{1}{x}$ , for n > 1.

$$\therefore \quad f_n'(x) = \begin{cases} nx^{n-1}\sin\frac{1}{x} - x^{n-2}\cos\frac{1}{x} & ,\text{when } x \neq 0 \\ 0 & ,\text{when } x = 0 \end{cases}$$ , where $n \in \mathbb{N}/\{1\}$ .

**3.** **(i)** $\lim_{x\to 0^-}\dfrac{1-2^{1/x}}{1+2^{1/x}} = \dfrac{1-2^{-\infty}}{1+2^{-\infty}} = \dfrac{1-0}{1+0} = 1$, $\lim_{x\to 0^+}\dfrac{1-2^{1/x}}{1+2^{1/x}} = \lim_{x\to 0^+}\dfrac{\frac{1}{2^{1/x}}-1}{\frac{1}{2^{1/x}}+1} = \dfrac{0-1}{0+1} = -1$

$\therefore$ $\lim_{x\to 0^-}\dfrac{1-2^{1/x}}{1+2^{1/x}} \neq \lim_{x\to 0^+}\dfrac{1-2^{1/x}}{1+2^{1/x}}$ and $\dfrac{1-2^{1/x}}{1+2^{1/x}}$ has a jump discontinuity.

**(ii)** $\lim_{x\to 0^-}\dfrac{1}{3^{1/x}+1} = \dfrac{1}{3^{-\infty}+1} = \dfrac{1}{0+1} = 1$, $\lim_{x\to 0^+}\dfrac{1}{3^{1/x}+1} = \lim_{x\to 0^+}\dfrac{1}{+\infty+1} = 0$

$\therefore$ $\lim_{x\to 0^-}\dfrac{1}{3^{1/x}+1} \neq \lim_{x\to 0^+}\dfrac{1}{3^{1/x}+1}$ and $\dfrac{1}{3^{1/x}+1}$ has a jump discontinuity.

**(iii)** $\lim_{x\to 0^-}\dfrac{x}{3^{1/x}+1} = \dfrac{x}{3^{-\infty}+1} = \dfrac{0}{0+1} = 0$, $\lim_{x\to 0^+}\dfrac{x}{3^{1/x}+1} = \lim_{x\to 0^+}\dfrac{0}{+\infty+1} = 0$

$\therefore$ $\lim_{x\to 0^-}\dfrac{x}{3^{1/x}+1} = \lim_{x\to 0^+}\dfrac{x}{3^{1/x}+1}$ and the function is undefined at x = 0 .

$\therefore$ $\dfrac{x}{3^{1/x}+1}$ has a removable discontinuity. Define $f(x) = \begin{cases} \dfrac{x}{3^{1/x}+1} & ,\text{if } x \neq 0 \\ 0 & ,\text{if } x = 0 \end{cases}$ .

Then f(x) is a continuous function with the discontinuity removed.

**4.** **(a)** $f'(0) = \lim_{h\to 0}\dfrac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\dfrac{e^{-\frac{1}{h^2}} - e^{-\frac{1}{0}}}{h} = \lim_{h\to 0}\dfrac{e^{-\frac{1}{h^2}}}{h} = \lim_{h\to 0}\dfrac{\frac{1}{h}}{e^{\frac{1}{h^2}}} = \lim_{h\to 0}\dfrac{\frac{1}{dh}\frac{1}{h}}{\frac{1}{dh}e^{\frac{1}{h^2}}}$ , by L'hospital rule.

$$= \lim_{h\to 0} \frac{-\frac{1}{h^2}}{\frac{2}{h^3} e^{\frac{1}{h^2}}} = \lim_{h\to 0} \frac{-h}{2e^{\frac{1}{h^2}}} = 0 \qquad \therefore \quad f'(x) = \begin{cases} \frac{2}{x^3} e^{-\frac{1}{x^2}} & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$$

**(b)** $f''(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{\frac{2}{h^3} e^{-\frac{1}{h^2}} - 0}{h} = \lim_{h\to 0} \frac{2e^{-\frac{1}{h^2}}}{h^4} = \lim_{h\to 0} \frac{\frac{2}{h^4}}{e^{\frac{1}{h^2}}} = \lim_{h\to 0} \frac{\frac{d}{dh}\frac{2}{h^4}}{\frac{d}{dh} e^{\frac{1}{h^2}}}$ , by L'hospital rule.

$$= \lim_{h\to 0} \frac{-\frac{8}{h^5}}{\frac{2}{h^3} e^{\frac{1}{h^2}}} = \lim_{h\to 0} \frac{-\frac{4}{h^2}}{e^{\frac{1}{h^2}}} = \lim_{h\to 0} \frac{-\frac{1}{dh}\frac{4}{h^2}}{\frac{1}{dh} e^{\frac{1}{h^2}}} = \lim_{h\to 0} \frac{\frac{8}{h^3}}{\frac{2}{h^3} e^{\frac{1}{h^2}}} = \lim_{h\to 0} \frac{4}{e^{\frac{1}{h^2}}} = 0$$

**5.** We need to show only : $f(x) = 0$ whenever $x$ is irrational .

$\forall\, x_0 \in \mathbf{I}$ and $x_0$ is irrational , there exists an infinite sequence $x_1, x_2, \ldots, x_n, \ldots$ of rational numbers such that $\lim_{n\to\infty} x_n = x_0$ .

Since $f(x)$ is continuous, $f(x_0) = f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} f(x_n) = 0$ , since $f(x_n) = 0$ , $x_n \in \mathbb{Q}$ .

**6.** **(a)** If $x_0 = ½$ , $f(½) = ½$ .

If $x$ is rational, $\lim_{x\to 1/2} f(x) = \lim_{x\to 1/2} x = \frac{1}{2}$ .

If $x$ is irrational, $\lim_{x\to 1/2} f(x) = \lim_{x\to 1/2} (1-x) = 1 - \frac{1}{2} = \frac{1}{2}$

$\therefore$ $x$ is continuous at $x_0 = ½$ .

If $x_0 \neq ½$ , and $x$ is rational. Then $f(x_0) = x_0$.

There exists an infinite sequence $x_1, x_2, \ldots, x_n, \ldots$ of irrational numbers such that $\lim_{n\to\infty} x_n = x_0$ .

Suppose that f(x) is continuous at that point, then $\lim_{x\to x_0} f(x) = f(x_0) = x_0$

But $\lim_{x\to x_0} f(x) = \lim_{n\to\infty} \lim_{x\to x_0} f(x_n) = \lim_{n\to\infty} (1 - x_n) = 1 - x_0$

This leads to contradiction as $1 - x_0 = x_0$ has no solution if $x_0 \neq ½$ .

The proof is similar if $x_0 \neq ½$ , and $x$ is irrational.

**(b)** If $g(x) = f(x)\, f(1 - x)$ , then $g(x) = x(1 - x)$ no matter $x$ is rational or irrational.

Obviously $g(x)$ is continuous since it is a quadratic function (polynomials are continuous)

**7.** $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{f(x)f(h) - f(x)}{h} = \lim_{h\to 0} \frac{f(x)(f(h)-1)}{h} = f(x) \lim_{h\to 0} \frac{f(h)-1}{h} = f(x) \lim_{h\to 0} \frac{hg(h)}{h}$

$$= f(x) \lim_{h\to 0} g(h) = f(x) \times 1 = f(x)$$

**8.** $f(xy) = f(x) + f(y) \quad \Rightarrow \quad f(1) = f(1\times 1) = f(1) + f(1) = 2f(1) \quad \Rightarrow \quad f(1) = 0.$

$$0 = f(1) = f(\frac{1}{x}x) = f\left(\frac{1}{x}\right) + f(x) \Rightarrow f\left(\frac{1}{x}\right) = -f(x)$$

$$f'(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{f(1+h)-0}{h} = \lim_{h\to 0}\frac{f(h)}{h}$$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{f(x+h)+f\left(\frac{1}{x}\right)}{h} = \lim_{h\to 0}\frac{f\left(\frac{x+h}{x}\right)}{h} = \lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right)}{h} = \frac{1}{x}\lim_{\frac{h}{x}\to 0}\frac{f\left(1+\frac{h}{x}\right)}{\frac{h}{x}} = \frac{f'(1)}{1}$$